International Journal of Solids and Structures 46 (2009) 1240–1250

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

Orthotropic elastic media having a closed form expression of the Green tensor

G. Bonnet *

Université Paris-Est, Laboratoire de Modélisation et Simulation, Multi-Echelle (CNRS FRE3160), 5 Boulevard Descartes, 77454 Marne la Vallée Cedex, France

ARTICLE INFO

Article history:
Received 5 August 2008
Received in revised form 16 October 2008
Available online 12 November 2008

Keywords:
Elasticity
Orthotropic
Cubic
Tetragonal
Green tensor
Anisotropic

ABSTRACT

Obtaining the Green tensor for the most general orthotropic medium is not generally possible in a closed form because the solution requires the roots of a sextic, often known as Stroh eigenvalues. The paper gives some conditions under which the sextic can be solved in a closed form for any direction within the space. It enables the construction of classes of orthotropic materials for which the Green tensor can be computed in a closed form (closed-form orthotropic or CFO) for any direction within the space. The cases of transversely isotropic, tetragonal and cubic materials are studied as special cases. The comparison between the exact Green function and approximate Green functions obtained from the nearest CFO material (in the sense of four different distances) is finally performed in the case of five examples of elasticity tensors.

1. Introduction

The Green tensor for elasticity in an infinite space is defined by the displacement field at any point within a linear elastic medium induced by a point force in any direction. It is the basis of many applications either to obtain the stress field due to defects (Mura, 1987) or to solve elasticity problems by integral equations methods. When the material is not isotropic, a fully explicit analytical solution for the Green function can be obtained for 2D problems. In the case of 3D problems a fully explicit expression of the Green tensor has been obtained only in some specific situations:

- for any direction within a transversely isotropic material (Kroner, 1953; Lejcek, 1969; Willis, 1969; Dahan and Predeleanu, 1980; Pan and Chou, 1976; Nakamura and Tanuma, 1997);
- for materials whose elastic tensors are obtained by linear transformation of the axes from transversely isotropic materials (Pouya and Zaoui, 2006; Pouya, 2007a,b) for materials characterized by the ellipsoidal anisotropy of De Saint Venant (1863);
- for orthotropic or anisotropic materials when the direction between the point where the displacement is computed and the point where the force is applied is parallel or perpendicular to some planes of symmetry (Ting and Lee, 1997; Lee, 2002).

Series solutions can also be obtained in other cases (Mura and Kinoshita, 1971; Mura, 1987; Chang and Chang, 1995; Kuznetsov, 1996; Faux and Pearson, 2000), but such solutions lead to computation times which could limit the possibility of applications. Approximate solutions can also be obtained for example in the case of cubic crystals (Dederichs and Leibfried, 1969).

For the general case of anisotropy, the solution can be put into the form of a scalar integral of a rational fraction (Lifshitz and Rozenzweig, 1947; Mura, 1987). Such a form of solution can be used for numerical purposes within the boundary element method by computing numerically the integral (Condat and Kirchner, 1987; Wang, 1997; Sales and Gray, 1998; Tonon et al., 2001; Lee, 2003). It needs, however, further developments and it induces a priori longer computation times than a closed form solution.

The main problem for obtaining a closed form of the Green tensor is that the denominator of the rational fraction which appears in the integral form of that tensor is a sixth order polynomial, whose roots cannot be obtained in a closed-form (Head, 1979) in the most general case. The purpose of the paper is to search elasticity tensors which display such a property and to show that in the case of some specific orthotropic material, the roots of the sixth order polynomial can be obtained for any direction of the space. The Green tensor can then be computed in a closed form for any direction of the space.

2. The Green tensor for an anisotropic material

The component $G_{km}(\mathbf{x}-\mathbf{y})$ of the Green tensor of an elastic medium is defined as the displacement component in the x_k-direction at point $\mathbf{x}$ when a unit body force in the x_m-direction is applied at point $\mathbf{y}$ in an infinitely extended media. These components comply to the equilibrium equations:

$$C_{ijks}\frac{\partial^2 G_{km}}{\partial x_j \partial x_s} + \delta_{im}\delta(\mathbf{x}-\mathbf{y}) = 0 \tag{1}$$

* Tel.: +33 1 60 95 72 20; fax: +33 1 60 95 77 99.
E-mail address: bonnet@univ-mlv.fr

0020-7683/$ - see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijsolstr.2008.10.033

where $\delta(\mathbf{x}-\mathbf{y})$ is the Dirac delta function, δ_{im} is the Kronecker delta and C_{ijks} are the elastic constants.

A classical derivation (Lifshitz and Rozenzweig, 1947; Mura, 1987) leads to the expression of the Green tensor $\mathbf{G}$ under the form of a contour integral:

$$\mathbf{G} = \frac{1}{8\pi^2 r}\int_C \mathbf{Q}^{-1}(\mathbf{k})\,ds(\mathbf{k}) \tag{2}$$

where r is the distance between point force and observation point.

The integrand is the inverse of the second order acoustic tensor $\mathbf{Q}$, whose components are the following:

$$Q_{ik}(\mathbf{k}) = C_{ijks}k_j k_s \tag{3}$$

The contour integral must be computed along the circle C of unit radius centered at the origin which is in the plane (P) perpendicular to the direction $\mathbf{x}-\mathbf{y}$.

If $\mathbf{y}$ is chosen at the origin, the cartesian coordinates of the unit vector in the direction $\mathbf{x}$ are given as functions of its spherical coordinates as follows:

$$(\sin(\phi)\cdot\cos(\theta), \sin(\phi)\cdot\sin(\theta), \cos(\phi))$$

Let $\mathbf{n}$ and $\mathbf{m}$ be two orthogonal unit vectors parallel to the plane (P); these two vectors can be chosen as follows:

- for $\mathbf{n}$: $(\sin(\theta), -\cos(\theta), 0)$;
- for $\mathbf{m}$: $(\cos(\phi)\cdot\cos(\theta), \cos(\phi)\cdot\sin(\theta), -\sin(\phi))$.

In the plane (P), the vector $\mathbf{k}$ can be expressed as:

$$\mathbf{k} = \cos\psi\,\mathbf{n} + \sin\psi\,\mathbf{m} = \cos\psi(\mathbf{n} + p\mathbf{m}) \tag{4}$$

where $p = tan(\psi)$

With these notations, Eq. (2) can be written as:

$$\mathbf{G} = \frac{1}{8\pi^2 r}\int_0^{2\pi} \mathbf{Q}^{-1}(\psi)\,d\psi \tag{5}$$

Let:

$$Q_{0ik} = C_{ijks}n_j n_s \qquad R_{ik} = C_{ijks}n_j m_s \qquad T_{ik} = C_{ijks}m_j m_s \tag{6}$$

The matrix $\mathbf{Q}(\psi)$ is a function of ψ which can be written:

$$\mathbf{Q}(\psi) = \mathbf{Q}_0\cos^2\psi + (\mathbf{R}+\mathbf{R}^T)\cos\psi\sin\psi + \mathbf{T}\sin^2\psi \tag{7}$$

$$= \mathbf{\Gamma}(p)\cos^2\psi \tag{8}$$

where

$$\mathbf{\Gamma}(p) = \mathbf{Q}_0 + p(\mathbf{R}+\mathbf{R}^T) + p^2\mathbf{T} \tag{9}$$

Finally, with the use of $p = \tan(\psi)$:

$$\mathbf{G} = \frac{1}{4\pi^2 r}\int_{-\infty}^{\infty}\mathbf{\Gamma}^{-1}(p)dp = \frac{1}{4\pi^2 r}\int_{-\infty}^{\infty}\frac{\widehat{\mathbf{\Gamma}}(p)}{|\mathbf{\Gamma}|(p)}\,dp \tag{10}$$

where $\widehat{\mathbf{\Gamma}}$ and $|\mathbf{\Gamma}|$ are the adjoint and the determinant of $\mathbf{\Gamma}$. The components of $\widehat{\mathbf{\Gamma}}$ are polynomials of fourth order and $|\mathbf{\Gamma}|$ is a sixth order polynomial.

Computing the integral in (10) by residue calculus requires the poles located at the roots of the sixth order polynomial $|\mathbf{\Gamma}|$ which are all complex (Ting, 1996). If these poles are known and if these poles are distinct, the Green tensor is given by:

$$\mathbf{G} = \frac{1}{2\pi r}i\sum_{v=1}^{3}\frac{\widehat{\mathbf{\Gamma}}(p_v)}{|\mathbf{\Gamma}|'(p_v)} \tag{11}$$

where p_v are the roots of $|\mathbf{\Gamma}|$ with a positive imaginary part and $|\mathbf{\Gamma}|'(p)$ is the derivative of $|\mathbf{\Gamma}|(p)$.

Obtaining the roots of $|\mathbf{\Gamma}|(p)$ is not generally possible by using radicals as it is well known from the work of Galois (Head, 1979). In the general case, it is possible (Ting, 1996) to obtain these values by computing numerically the eigenvalues of the matrix

$$\begin{bmatrix} -\mathbf{T}^{-1}\mathbf{R}^T & \mathbf{T}^{-1} \\ \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^T - \mathbf{Q_0} & -\mathbf{R}(\mathbf{T}^T)^{-1} \end{bmatrix} \tag{12}$$

In some special cases, the equation $|\mathbf{\Gamma}|(p) = 0$ can be solved in a closed form; for example if the symmetry is such that the odd powers of p are cancelled in $\Gamma(p)$. In such a case, the equation is a third order equation in p^2. The equation $|\mathbf{\Gamma}|(p) = 0$ has, however, no closed form solution in the most general case. The aim of the following is to describe situations, in the case of orthotropic materials only, where the roots of $|\mathbf{\Gamma}|(p)$ can be obtained in a closed form for any direction of the vector $\mathbf{x}-\mathbf{y}$.

3. Principles for the factorization of the determinant of the acoustic tensor

The previous section has shown that computing the integral in (10) by residue calculus requires the computation of the poles located at the roots of the sixth order polynomial $|\mathbf{\Gamma}|(p)$. The properties of $|\mathbf{\Gamma}|(p)$ are, however, closely related to the properties of the determinant $\Delta(\mathbf{k}) = |\mathbf{Q}(\mathbf{k})|$ of the acoustic tensor $\mathbf{Q}(\mathbf{k})$.

In the following, the material will be assumed orthotropic and all C_{ijkl} are, in the axes of symmetry of the material, functions of nine constants which can be denoted c_{11}, c_{22}, c_{33}, c_{23}, c_{31}, c_{12}, c_{44}, c_{55}, c_{66} where the classical notation with two indices c_{ij} is used:

$$c_{ij} = C_{iijj} \quad \text{for} \quad i = 1,\ldots,3,\ j = 1,\ldots,3$$

$$c_{II} = C_{ijij} \quad \text{for} \quad I = 4,\ i = 2,\ j = 3 \quad \text{or} \quad I = 5,\ i = 3,\ j = 1 \quad \text{or} \\ I = 6,\ i = 1,\ j = 2$$

$$c_{IJ} = 0 \quad \text{for} \quad I \geqslant 4, J \leqslant 3 \quad \text{or} \quad J \geqslant 4,\ I \leqslant 3 \quad \text{or} \\ (I,J \geqslant 4 \quad \text{and} \quad I \neq J)$$

This change of notation assumes that the new matrix coefficients allow the computing of the components of the stress tensor from the components of $\gamma_{ij} = 2\epsilon_{ij}$. This leads to the matrix related to the acoustic tensor given in Appendix D.

If the coordinate axes are chosen along the axes of symmetry of the material, the determinant $\Delta(\mathbf{k})$ of the matrix related to the acoustic tensor is an homogeneous function of third order of the squares of the coordinates k_1, k_2, k_3 of $\mathbf{k}$ given by:

$$\Delta = a_{111}l_1^3 + a_{222}l_2^3 + a_{333}l_3^3 + a_{112}l_1^2 l_2 + a_{113}l_1^2 l_3 + a_{221}l_2^2 l_1 \\ + a_{223}l_2^2 l_3 + a_{331}l_3^2 l_1 + a_{332}l_3^2 l_2 + a_1 l_1 l_2 l_3 \tag{13}$$

where $l_i = k_i^2$ and where the coefficients a_{ijk} are functions of the elastic coefficients c_{IJ} given in Appendix A.

As explained previously, the case of a transversely isotropic material (or the one of a material obtained from a transversely isotropic material by a scaling of the axes) is such that the Stroh eigenvalues can be obtained in a closed form for the 3D case, allowing the Green tensor to be obtained in a closed form.

It is easy to show that such a result is due to the fact that the determinant Δ of the acoustic tensor can be factorized by using homogeneous order 2 polynomials in k_i (or linear homogeneous polynomials in l_i).

Indeed, for a transversely isotropic material, the components of the elasticity tensor can be written as functions of five elastic constants $c_{11}, c_{33}, c_{12}, c_{13}, c_{44}$, other constants being given by:

$$\begin{aligned} c_{22} &= c_{11} \\ c_{23} &= c_{13} \\ c_{55} &= c_{44} \\ c_{66} &= \frac{1}{2}(c_{11} - c_{12}) \end{aligned} \tag{14}$$

The coefficients a_{ijk} of the polynomial $\Delta(l_1, l_2, l_3)$ are then given by the expressions given in Appendix B.

Finally, the polynomial $\Delta(l_1, l_2, l_3)$ can be written as follows:

$$\Delta = A_3(l_1+l_2)^3 + A_2(l_1+l_2)^2 l_3 + A_1(l_1+l_2)l_3^2 + A_0 l_3^3 \tag{15}$$

where the coefficients A_i are given by:

$$A_3 = \frac{1}{2}c_{11}c_{44}(c_{11}-c_{12}) \tag{16}$$

$$A_2 = c_{11}c_{44}^2 + \frac{1}{2}c_{11}^2c_{33} - \frac{1}{2}c_{13}^2c_{11} - \frac{1}{2}c_{11}c_{12}c_{33} + \frac{1}{2}c_{13}^2c_{12} - c_{13}c_{11}c_{44} + c_{13}c_{12}c_{44} \tag{17}$$

$$A_1 = -c_{44}c_{13}^2 - 2c_{44}^2c_{13} + \frac{3}{2}c_{44}c_{11}c_{33} - \frac{1}{2}c_{44}c_{12}c_{33} \tag{18}$$

$$A_0 = c_{44}^2c_{33} \tag{19}$$

Then, the polynomial Δ is:

$$\Delta = A_3(l_1+l_2-r_1l_3)(l_1+l_2-r_2l_3)(l_1+l_2-r_3l_3) \tag{20}$$

where r_1, r_2, r_3 are the solutions of the third order equation:

$$A_3r^3 + A_2r^2 + A_1r + A_0 = 0 \tag{21}$$

One solution is given by:

$$r_1 = -2\frac{c_{44}}{c_{11}-c_{12}} = -\frac{c_{44}}{c_{66}} \tag{22}$$

while r_2 and r_3 are solutions of the quadratic equation:

$$c_{44}c_{33} + (c_{11}c_{33} - 2c_{13}c_{44} - c_{13}^2)r + c_{11}c_{44}r^2 = 0 \tag{23}$$

Then, the eigenvalues are obtained by solving the three quadratic equations in p obtained from the factorization (20) by replacing l_k by $(n_k + pm_k)^2$.

Each equation is given by:

$$1 + p^2\left(\cos^2(\phi) - r_k\sin^2(\phi)\right) = 0 \tag{24}$$

The eigenvalues are the roots with a positive imaginary part given by:

$$p_k = \frac{i}{\sqrt{\cos^2\phi - r_k\sin^2\phi}} \tag{25}$$

A comparison can be made with eigenvalues obtained by Ting and Lee (1997) by noticing that the expression of the eigenvalues depends of the choice of $\mathbf{m}$ and $\mathbf{n}$. Taking $\mathbf{k} = \cos\psi\mathbf{m} - \sin\psi\mathbf{n}$ instead of $\mathbf{k} = \cos\psi\mathbf{n} + \sin\psi\mathbf{m}$ leads to:

$$p^2 + \cos^2(\phi) - r_k\sin^2(\phi) = 0 \tag{26}$$

instead of (24).

Finally, for the particular value of r_k given by (22), it leads to:

$$p^2c_{66} + \cos^2(\phi)c_{66} + \sin^2(\phi)c_{44} = 0 \tag{27}$$

which is obtained in Ting and Lee (1997).

Such a result recovers the well-known result along which there is a closed form expression of the Green tensor for a transversely isotropic material for any direction $\mathbf{x}-\mathbf{y}$, showing that the eigenvalues do not depend on θ, as a consequence of the symmetry of the material.

In addition, this result explains why the solution can be obtained for any direction of the space: it is the factorization in polynomials in k_i which is the key to obtain eigenvalues in a closed form for any direction.

Using this idea, it is natural to seek factorizations of Δ which can be of two kinds:

- factorization into one quadratic homogeneous polynomial in l_i and a linear and homogeneous function of l_i;
- factorization into three linear and homogeneous polynomials in l_i.

Starting by the first kind, it is easy to show that *if* the function $\Delta(l_1, l_2, l_3)$ can be written as the product of a quadratic homogeneous function of l_i and a linear homogeneous function of l_i it is possible to obtain a closed form expression of the roots of Δ (for any direction) $\mathbf{x}-\mathbf{y}$.

Indeed, taking into account the higher order term in l_1 within (13), let us assume that Δ can be decomposed into the following form:

$$\Delta = a_{111}P_4(l_1, l_2, l_3)\cdot P_2(l_1, l_2, l_3) \tag{28}$$

where P_2 and P_4 are given by:

$$P_4 = \left(l_1^2 + b_{22}l_2^2 + b_{33}l_3^2 + b_{12}l_1l_2 + b_{23}l_2l_3 + b_{31}l_1l_3\right) \tag{29}$$

$$P_2 = (l_1 + c_2l_2 + c_3l_3) \tag{30}$$

where b_{ij} and c_k are the coefficients of the most general homogeneous polynomials of orders 2 and 1 in l_i which have the term of highest power in l_1 equal to 1. These coefficients (like for the case of transverse isotropy) should depend only on the elastic coefficients.

If such a decomposition is possible, the function $\Delta(k_1, k_2, k_3)$ can be written:

$$\Delta(k_1,k_2,k_3) = a_{111}\left(k_1^4 + b_{22}k_2^4 + b_{33}k_3^4 + b_{12}k_1^2k_2^2 + b_{23}k_2^2k_3^2 + b_{31}k_1^2k_3^2\right)\cdot\left(k_1^2 + c_2k_2^2 + c_3k_3^2\right) \tag{31}$$

Replacing k_i into the definition of $|\mathbf{\Gamma}|$ by its expression given by (4) leads to:

$$|\mathbf{\Gamma}| = a_{111}Q_4(p)\cdot Q_2(p) \tag{32}$$

where $Q_4(p)$ and $Q_2(p)$ are the polynomials:

$$Q_4(p) = (n_1 + m_1p)^4 + b_{22}(n_2 + m_2p)^4 + b_{33}(n_3 + m_3p)^4 + b_{12}(n_1 + m_1p)^2(n_2 + m_2p)^2 + b_{23}(n_2 + m_2p)^2(n_3 + m_3p)^2 + b_{31}(n_3 + m_3p)^2(n_1 + m_1p)^2 \tag{33}$$

$$Q_2(p) = (n_1 + m_1p)^2 + c_2(n_2 + m_2p)^2 + c_3(n_3 + m_3p)^2 \tag{34}$$

where n_i and m_i are the components of the unit vectors $\mathbf{n}$ and $\mathbf{m}$ defined in Section 2, which are perpendicular to direction $\mathbf{x}-\mathbf{y}$.

The decomposition of Δ defined in (32)–(34) ensures therefore that the poles of the rational fraction $\Gamma^{-1}(p)$ can be computed in a closed form by solving the quadratic and fourth order equations $Q_4(p) = 0$ and $Q_2(p) = 0$, where Q_4 and Q_2 are obtained from (33) and (34). Such a decomposition was used for example by Ting and Lee (1997) for the Green tensor at points located on the normal to the symmetry plane of a monoclinic material.

When such a decomposition is possible, the elasticity tensor will be called thereafter "Closed Form Orthotropic" tensor of class 4 (order 4 is the higher degree of the polynomials used in the decomposition), abbreviated thereafter by "CFO4 class". The related material will be called "CFO4 material".

The case which comprises three linear polynomials in l_k is such that:

$$\Delta = a_{111}P_a(l_1, l_2, l_3)\cdot P_b(l_1, l_2, l_3)\cdot P_c(l_1, l_2, l_3) \tag{35}$$

where

$$P_a = l_1 + a_2l_2 + a_3l_3 \tag{36}$$

and similar definitions for P_b and P_c by replacing a_i $(i = 2, 3)$ by b_i and c_i, where all coefficients depend only on the elastic coefficients. The elasticity tensors which lead to determinants which can be put into such a form will be called thereafter of "CFO2 class". Such elasticity tensors have less symmetries than a transversely isotropic tensor (or than tensors obtained from transversely isotropic tensors by transformation of axes), because it involves more independent parameters, as it will be shown thereafter.

4. Factorization of the determinant of the acoustic tensor for the CFO4 class

We look now at the conditions under which a decomposition such as the one defined by (28)–(30) is possible.

Identifying (13) and 28, 29, 30 leads to:

$$\begin{aligned} d_{222} &= b_{22}c_2 \\ d_{333} &= b_{33}c_3 \\ d_{112} &= c_2 + b_{12} \\ d_{113} &= c_3 + b_{13} \\ d_{221} &= b_{22} + b_{12}c_2 \\ d_{331} &= b_{33} + b_{13}c_3 \\ d_{223} &= b_{23}c_2 + b_{22}c_3 \\ d_{332} &= b_{23}c_3 + b_{33}c_2 \\ d_{123} &= b_{23} + b_{13}c_2 + b_{12}c_3 \end{aligned} \tag{37}$$

where the coefficients d_{ijk} are functions of the elastic coefficients obtained from the coefficients d_{ijk} by:

$$d_{ijk} = a_{ijk}/a_{111} \tag{38}$$

Combining these equations leads to four equations in c_2 and c_3:

Two equations of the third order in c_2 and c_3

$$d_{222} = c_2^3 - d_{112}c_2^2 + d_{221}c_2 \tag{39}$$

$$d_{333} = c_3^3 - d_{113}c_3^2 + d_{331}c_3 \tag{40}$$

Two coupled equations:

$$d_{223} = c_2 d_{123} - c_2^2 d_{113} + 3c_3c_2^2 - 2c_3c_2d_{112} + c_3d_{221} \tag{41}$$

$$d_{332} = c_3 d_{123} - c_3^2 d_{112} + 3c_2c_3^2 - 2c_2c_3d_{113} + c_2d_{331} \tag{42}$$

These four relations involving c_2 and c_3 are in general independent and they have a solution only if suitable relations between the elastic coefficients of the elasticity tensor are met. However, it will be shown thereafter that the compatibility conditions and the expression of the polynomials obtained by decomposition become simpler by using suitable transformations of the axes.

It was shown indeed by Pouya and Zaoui (2006) that suitable transformations of the coordinate system may be used to simplify some problems related to the computation of the Green tensor.

Let us call $\mathbf{x}$ the column containing the coordinates of the position vector in the original coordinate system, $\mathbf{x}'$ containing the coordinates of the position vector in the transformed system, with similar notations $\mathbf{u}$ and $\mathbf{u}'$ for the vector displacement.

The transformation matrix $\mathbf{P}$, assumed invertible, defines the transformation of the coordinates of the position vector.

$$\mathbf{x} = \mathbf{P}\cdot\mathbf{x}' \tag{43}$$

Pouya and Zaoui (2006) have shown that the displacement vector must be transformed by

$$\mathbf{u} = \mathbf{S}\cdot\mathbf{u}' \tag{44}$$

where

$$\mathbf{S} = (\mathbf{P}^T)^{-1} \tag{45}$$

where $\mathbf{P}^T$ is the transpose of $\mathbf{P}$.

Under these conditions, the elasticity tensor is transformed into:

$$C'_{mnpq} = C_{ijkl}S_{im}S_{jn}S_{kp}S_{lq} \tag{46}$$

and the Green tensor is transformed by:

$$\mathbf{G}' = |\mathbf{P}|\mathbf{P}^T\cdot\mathbf{G}\cdot\mathbf{P} \tag{47}$$

Using such transformations can simplify significantly the conditions (39)–(42).

To keep the symmetry of the material with respect to the axes, a scaling will be used in the following, such as $x'_i = \alpha_i x_i$ leading to the transformed values of the coordinates of the wave vector $k_i = \alpha_i k'_i$. Such a transformation leads obviously to diagonal matrices $\mathbf{P}$ and $\mathbf{S}$.

In the new system of axes, the coefficients d_{ijk} of the previous equations are transformed into d'_{ijk} and the coefficients c_i and b_{ij} are transformed into c'_i and b'_{ij}.

The compatibility equations within the new system of axes are similar to (39)–(42).

Using the relation between d'_{ijk} and d_{ijk} leads then to:

$$d_{222}\beta_2^3 = \beta_1^3c_2'^3 - d_{112}\beta_1^2\beta_2c_2'^2 + d_{221}\beta_2^2\beta_1c'_2 \tag{48}$$

$$d_{333}\beta_3^3 = \beta_1^3c_3'^3 - d_{113}\beta_1^2\beta_3c_3'^2 + d_{331}\beta_3^2\beta_1c'_3 \tag{49}$$

where $\beta_i = \alpha_i^2$ and where c'_2 and c'_3 are the values of c_2 and c_3 in the new coordinate system.

Dividing the first one by β_2^3 and the second one by β_3^3 leads to:

$$d_{222} = r_2^3 - d_{112}r_2^2 + d_{221}r_2 \tag{50}$$

$$d_{333} = r_3^3 - d_{113}r_3^2 + d_{331}r_3 \tag{51}$$

with $r_i = \frac{\beta_1 c'_i}{\beta_i}$.

It means that the values of c'_2 and c'_3 can be adjusted to any preset value by choosing conveniently the scaling of the axes.

Let us choose $c'_2 = c'_3 = 1$. It implies $\beta_i = \beta_1/r_i$, where r_i $(i = 2, 3)$ are real and positive solutions of Eqs. (50) and (51). Due to the fact that d_{222} and d_{333} are positive, there are always such real positive roots.

The following choice of scaling can be used: $\beta_1 = 1, \beta_2 = 1/r_2$, $\beta_3 = 1/r_3$.

In the new system of axes, Eqs. (41) and (42) become:

$$d'_{223} = d'_{123} - d'_{113} + 3 - 2d'_{112} + d'_{221} \tag{52}$$

$$d'_{332} = d'_{123} - d'_{112} + 3 - 2d'_{113} + d'_{331} \tag{53}$$

These two equations display the conditions, in the new system of coordinates, between the elastic constants of the orthotropic material which must be satisfied in order to ensure that the elastic material will be of CFO4 class. From a general point of view, the elasticity tensor of an orthotropic material depends on nine independent constants and the compatibility conditions imply that the elastic parameters of a CFO4 material depend on less than nine constants.

In the transformed system, the polynomials P_4 and P_2 which appear in the factorized form, if it exists, are finally given by:

$$\begin{aligned} P_4 = {} & k_1'^4 + d'_{222}k_2'^4 + d'_{333}k_3'^4 + (d'_{112}-1)k_1'^2k_2'^2 + (d'_{113}-1)k_1'^2k_3'^2 \\ & + (d'_{223} - d'_{222})k_2'^2k_3'^2 \end{aligned} \tag{54}$$

$$P_2 = k_1'^2 + k_2'^2 + k_3'^2 \tag{55}$$

where k'_i are the components of the wave numbers along the transformed axes.

5. Factorization of the determinant of the acoustic tensor for the CFO2 class

Similar operations can be performed in the case of the CFO2 class. In this case, there are still nine equations analogous to (37) to solve, but the factorization leads to one more compatibility equation. These equations can be obtained as for the CF04 class. In a first step, elimination of a_1, b_1, a_2, b_2 from identification between Δ and its factorized form leads to Eqs. (39)–(42) and to a complementary equation

$$(2d_{123} - s_2 s_3 - 2c_3 s_2 - 2c_2 s_3)^2 - (s_2^2 - 4t_2)(s_3^2 - 4t_3) = 0 \tag{56}$$

where $t_2,\ t_3,\ s_2,\ s_3$ are defined as follows:

$$t_k = d_{kkk}/c_k \tag{57}$$
$$s_k = d_{k11} - c_k \tag{58}$$

The same transformation of axes as the one used in the previous section allows to fix c'_2 and c'_3 at 1, leading to the compatibility Eqs. (52) and (53) and to a complementary compatibility equation:

$$(2d'_{123} - s'_2 s'_3 - 2s'_2 - 2s'_3)^2 - (s'^2_2 - 4t'_2)(s'^2_3 - 4t'_3) = 0 \tag{59}$$

where d'_{ijk} are, as in the previous section, the values of d_{ijk} in the transformed basis, t'_k and s'_k being given by:

$$t'_k = d'_{kkk} \tag{60}$$
$$s'_k = d'_{k11} - 1 \tag{61}$$

The polynomials $P_a,\ P_b,\ P_c$, which appear in the factorized form of Δ are given, in the transformed axes by:

$$P_a = k'^2_1 + A_2 k'^2_2 + A_3 k'^2_3 \tag{62}$$
$$P_b = k'^2_1 + B_2 k'^2_2 + B_3 k'^2_3 \tag{63}$$
$$P_c = k'^2_1 + k'^2_2 + k'^2_3 \tag{64}$$

where A_2 and B_2 are solutions of the second order equation:

$$X^2 - (d'_{221} - 1)X + d'_{222} = 0 \tag{65}$$

while A_3 and B_3 are solutions of the second order equation:

$$X^2 - (d'_{331} - 1)X + d'_{333} = 0 \tag{66}$$

6. Study of materials with classical symmetries

Within the framework of the previous section, it is now possible to study materials having more symmetries than the most general symmetry of an orthotropic material and to check if such materials are of any CFO class or if there is a compatibility condition ensuring that the material is of a CFO class.

6.1. Transversely isotropic material

As shown in Section 2, a transversely isotropic material is of the CFO2 class, according to our previous definition. By scaling of the axes as the one used in the previous section, the homogeneous polynomials of the previous section can be put into the following form:

$$P_a = k'^2_1 + k'^2_2 + A_3 k'^2_3 \tag{67}$$
$$P_b = k'^2_1 + k'^2_2 + B_3 k'^2_3 \tag{68}$$
$$P_c = k'^2_1 + k'^2_2 + k'^2_3 \tag{69}$$

It is obvious that only a scaling along x_3 is necessary to obtain such a decomposition. This form involves two parameters A_3 and B_3 instead of four in the most general case of a CFO2 material. It means that the most general CFO2 material has less symmetries than a transversely isotropic material or a material obtained from a transversely isotropic material by scaling of the axes.

6.2. Tetragonal (and orthotropic) material

6.2.1. Case of CFO4 class

For a tetragonal (and orthotropic) material, the elasticity tensor is defined by six parameters, because the symmetry induces the following relations:

$$\begin{aligned} c_{22} &= c_{11} \\ c_{55} &= c_{44} \\ c_{23} &= c_{13} \end{aligned} \tag{70}$$

Under these conditions, the polynomial Δ can be written for a CFO4 material as:

$$\begin{aligned} \Delta = {} & a_{111}(l_1^3 + l_2^3) + a_{333} l_3^3 + a_{113}(l_1^2 + l_2^2) l_3 + a_{331}(l_1 + l_2) l_3^2 \\ & + a_{112}(l_1 + l_2) l_1 l_2 + a_{123} l_1 l_2 l_3 \end{aligned} \tag{71}$$

where the coefficients a_{ijk} are given in Appendix C. Taking into account the symmetries, the factorization must be effected under the form:

$$\Delta = a_{111}(l_1^2 + l_2^2 + b_{33} l_3^2 + b_{12} l_1 l_2 + b_{13}(l_1 + l_2) l_3) \cdot (l_1 + l_2 + c_3 l_3) \tag{72}$$

The relations between the coefficients of the polynomials can be given by using the coefficients $d_{ijk} = a_{ijk}/a_{111}$:

$$\begin{aligned} d_{333} &= c_3 b_{33} \\ d_{112} &= b_{12} + 1 \\ d_{113} &= b_{13} + c_3 \\ d_{331} &= c_3 b_{13} + b_{33} \\ d_{123} &= 2b_{13} + c_3 b_{12} \end{aligned} \tag{73}$$

Eliminating the coefficients b_{12}, b_{13}, b_{33} leads to Eq. (40) and to the following equation:

$$d_{123} = 2d_{113} - 3c_3 + c_3 d_{112} \tag{74}$$

This equation can be obtained from Eqs. (41) and (42) in the general case of orthotropy by putting $c_2 = 1,\ b_{22} = 1,\ b_{23} = b_{13},\ d_{113} = d_{223}$ and $d_{112} = d_{221}$. As in the previous section, a suitable transformation of axes allows to ensure the equivalent of Eq. (74) with $c'_3 = 1$ (c'_3 being the transformed of c_3 by a convenient scaling). In the transformed system, there remains one compatibility equation, which can be written:

$$d'_{123} = 2d'_{113} - 3 + d'_{112} \tag{75}$$

where d'_{ijk} are computed from the transformed elastic constants as previously.

It is easy to check that the case of the transversely isotropic medium leads to coefficients d'_{ijk} which comply to Eq. (75).

It is, however, possible to construct tetragonal materials which are not transversely isotropic and for which the compatibility equation related to the CFO4 property is met.

6.2.2. Case of CFO2 class

Taking into account the symmetries of the material, a tetragonal material which is of the CFO2 class is characterized by the determinant Δ given by:

$$\Delta = a_{111}(l_1 + l_2 + a_3 l_3)(l_1 + l_2 + b_3 l_3)(l_1 + l_2 + c_3 l_3) \tag{76}$$

This form is identical to the one given by (20), which shows that a tetragonal material of CFO2 class is necessarily transversely isotropic.

6.3. Cubic material

In the case of a cubic material, the components of the elasticity tensor are constrained by:

$$\begin{aligned} c_{11} &= c_{22} = c_{33} \\ c_{12} &= c_{23} = c_{31} \\ c_{44} &= c_{55} = c_{66} \end{aligned} \tag{77}$$

The polynomial Δ can then be written as:

$$\Delta = a_{111}\left(l_1^3 + l_2^3 + l_3^3\right) + a_{112}\left(l_1^2 l_2 + l_1^2 l_3 + l_2^2 l_1 + l_2^2 l_3 + l_3^2 l_1 + l_3^2 l_2\right) + a_{123} l_1 l_2 l_3 \quad (78)$$

where

$$\begin{aligned} a_{111} &= c_{11}c_{44}^2 \\ a_{112} &= c_{11}^2 c_{44} + c_{44}^2 c_{11} - 2c_{12}c_{44}^2 - c_{12}^2 c_{44} \\ a_{123} &= -3c_{11}c_{12}^2 - 6c_{11}c_{12}c_{44} + 6c_{12}^2 c_{44} + 6c_{12}c_{44}^2 + 2c_{12}^3 + c_{11}^3 + 4c_{44}^3 \end{aligned} \quad (79)$$

Taking into account the symmetries of the material, the factorization can be written as:

$$\Delta = a_{111}(l_1 + l_2 + l_3)\left(l_1^2 + l_2^2 + l_3^2 + b_{12}(l_1 l_2 + l_2 l_3 + l_3 l_1)\right) \quad (80)$$

The relations between the coefficients are:

$$\begin{aligned} d_{123} &= 3b_{12} \\ d_{112} &= b_{12} + 1 \end{aligned} \quad (81)$$

Finally, the compatibility condition can be written as:

$$c_{11}^3 + 4c_{44}^3 + 2c_{12}^3 - 3c_{11}c_{12}^2 - 6c_{11}c_{12}c_{44} + 9c_{12}^2 c_{44} + 12c_{12}c_{44}^2 - 3c_{11}^2 c_{44} = 0 \quad (82)$$

The roots of that equation in c_{44} are $c_{44} = -c_{11} - 2c_{12}$ and $c_{44} = (c_{11} - c_{12})/2$(double).

The first solution being not valid, it implies that only the double solution, which corresponds to the case of isotropy, can be used. It means that there is not any possibility to obtain a factorization of Δ in the case of a cubic, not isotropic material. The only cubic materials which are of CFO2 or CFO4 class are isotropic.

7. Components of the Green tensor for a CFO material

7.1. Eigenvalues for the case of a CFO4 material

The eigenvalues p_v are obtained from the solution of equations $P_4 = 0$ and $P_2 = 0$ where P_4 and P_2 are given in Section 4.

The quartic leading to the eigenvalues is a function of the angles ϕ and θ given by:

$$f_4 p^4 + f_3 p^3 + f_2 p^2 + f_1 p + f_0 = 0 \quad (83)$$

where the coefficients of the polynomial are given by:

$$f_4 = (cp^2 st^2 B_{22} + sp^2 B_{23} + B_{12})cp^2 st^2 + 1 + sp^2 B_{13} \quad (84)$$

$$f_3 = 2\left[(-2st^2 B_{22} + B_{23})cp^2 - B_{12} - B_{23}\right] st \cdot ct \cdot cp \quad (85)$$

$$f_2 = 6ct^2 cp^2 st^2 B_{22} + (sp^2 B_{33} - st^2 B_{12} + B_{13} + B_{23} ct^2) sp^2 + B_{12} + 2 \quad (86)$$

$$f_1 = -2(2ct^2 B_{22} + B_{12}) st \cdot ct \cdot cp \quad (87)$$

$$f_0 = ct^4 B_{22} + 1 + ct^2 B_{12} \quad (88)$$

where the compact notations for the trigonometric functions $sp = \sin(\phi)$, $cp = \cos(\phi)$, $st = \sin(\theta)$, $ct = \cos(\theta)$ are used and where B_{ij} are functions of the coefficients d'_{ijk} given below. The coefficients d'_{ijk} depend only on the elastic constants in the transformed system of axes by relations of Appendix A.

$$B_{kk} = 2 + d'_{kkk} - d'_{11k} \quad (89)$$

$$B_{23} = 4 - d'_{112} - d'_{113} + d'_{223} - d'_{222} \quad (90)$$

$$B_{1k} = d'_{11k} - 3 \quad (91)$$

Solutions of Eq. (83) can be obtained by the classical Ferrari's solution.

The quadratic equation leading to the last eigenvalue becomes, in the transformed system of axes:

$$1 + p^2 = 0 \quad (92)$$

This leads to the constant eigenvalue $p = i$, which does not depend on the direction of the axes (in the transformed system of coordinates). As shown thereafter, the fact that $p = i$ is an eigenvalue for all direction in the transformed axes is a property which is shared by the CFO2 material.

An elastic elasticity tensor is therefore of a CFO class if the transformation of axes defined by $\beta_1 = 1$ and $\beta_i = r_i$, r_i being the solutions of Eqs. (50) and (51) leads to a common eigenvalue $p = i$ for any direction between source and observation point.

It is obvious from the expressions of f_1 and f_3 that these coefficients are null when $\phi = 0$, $\theta = 0$ or $\theta = \pi$. In these cases, the direction of the observation point is along one of the planes of symmetry and it is known that the determinant of the acoustic tensor becomes in this case a function of p^2.

7.2. Eigenvalues for the case of a CFO2 material

In this case, the form of the polynomials P_a, P_b and P_c show that there is still a constant eigenvalue $p = i$ in the transformed system of coordinates. The other eigenvalues are solutions of:

$$[cp^2(ct^2 + A_2 st^2) + A_3 sp^2]p^2 + 2st \cdot cp \cdot ct(1 - A_2)p + st^2 + A_2 ct^2 = 0 \quad (93)$$

and a similar equation replacing A_k by B_k, the coefficients A_k and B_k being coefficients of the elastic constants which are solutions of Eqs. (65)–(67)

In the case of a transversely isotropic material, $A_2 = B_2 = 1$, and Eqs. (67)–(69) are recovered. Another case of interest is the case $B_2 = B_3 = 1$. In this case, the eigenvalue $p = i$ is double for any direction of the axes (similarly to the case of isotropy where $p = i$ is a triple eigenvalue).

7.3. Computation of the Green tensor with known eigenvalues

When the eigenvalues are obtained, the components of the Green tensor are obtained, when all eigenvalues are distinct, from the relations given by

$$\mathbf{G} = \frac{1}{4\pi r} 2i \sum_{v=1}^{3} \frac{\widehat{\mathbf{\Gamma}}(p_v)}{|\mathbf{\Gamma}(p_v)|'} \quad (94)$$

where $\widehat{\mathbf{\Gamma}}$ is the adjoint of $\mathbf{\Gamma}$ and $|\mathbf{\Gamma}|'$ the derivative of its determinant with respect to p, their value being computed for the three eigenvalues p_v $(v = 1.3)$.

The terms $|\mathbf{\Gamma}(p_v)|'$ are obtained from the eigenvalues as follows: $|\mathbf{\Gamma}(p_1)|'$ is given by Ting and Lee (1997):

$$|\mathbf{\Gamma}(p_1)|' = 2i|\mathbf{T}|\beta_1(p_1 - p_2)(p_1 - \overline{p_2})(p_1 - p_3)(p_1 - \overline{p_3}) \quad (95)$$

where β_1 is the (positive) imaginary part of p_1 and $\overline{p_v}$ the complex conjugate of p_v.

Similar expressions are obtained for the terms related to p_2 and p_3 in the previous relation.

Following Ting and Lee (1997), each cofactor is a polynomial of degree four in p which can be expanded as:

$$\widehat{\mathbf{\Gamma}}(p) = \sum_{n=0}^{4} p^n \widehat{\mathbf{\Gamma}}^{(n)} \quad (96)$$

where the matrices $\hat{\mathbf{\Gamma}}^{(n)}$ are independent of p_v.

These matrices depend only of the elasticity tensor and of the components of the vectors $\mathbf{n}$ and $\mathbf{m}$ through the matrices $\mathbf{Q_0}$, $\mathbf{T}$ and $\mathbf{R}_1 = \mathbf{R} + \mathbf{R}^T$. They are given below as functions of the matrices of cofactors of $\mathbf{Q_0}$, $\mathbf{R}_1$, $\mathbf{T}$, $\mathbf{Q_0} + \mathbf{R}_1$, $\mathbf{T} + \mathbf{R}_1$, $\mathbf{T} + \mathbf{Q_0}$

$$\widehat{\Gamma}^{(4)} = \widehat{\mathbf{T}} \tag{97}$$

$$\widehat{\Gamma}^{(3)} = \{\mathbf{T}, \mathbf{R_1}\} \tag{98}$$

$$\widehat{\Gamma}^{(2)} = \{\mathbf{T}, \mathbf{Q_0}\} + \widehat{\mathbf{R}}_1 \tag{99}$$

$$\widehat{\Gamma}^{(1)} = \{\mathbf{Q_0}, \mathbf{R_1}\} \tag{100}$$

$$\widehat{\Gamma}^{(0)} = \widehat{\mathbf{Q_0}} \tag{101}$$

where $\widehat{\mathbf{A}}$ is the matrix of the cofactors of $\mathbf{A}$ and where the expression $\{\mathbf{A},\mathbf{B}\}$ means:

$$\{\mathbf{A}, \mathbf{B}\} = \widehat{\mathbf{A}+\mathbf{B}} - \widehat{\mathbf{B}} - \widehat{\mathbf{B}} \tag{102}$$

Finally, the Green matrix $\mathbf{G}$ is given by:

$$\mathbf{G} = \frac{1}{4\pi r|\mathbf{T}|}\sum_{n=0}^{4} q_n \widehat{\Gamma}^{(n)} \tag{103}$$

where q_n is given by:

$$q_n = q_{n(1)} + q_{n(2)} + q_{n(3)} \tag{104}$$

$$q_{n(1)} = \frac{p_1^n(p_2 - \overline{p_2})(p_3 - \overline{p_3})}{\beta_1(p_1 - p_2)(p_1 - \overline{p_2})(p_1 - p_3)(p_1 - \overline{p_3})} \tag{105}$$

with similar expressions for $q_{n(2)}$ and $q_{n(3)}$.

The main problem with such an expression is that it does not work when there are double roots, which corresponds to the "degenerate" case (Wang and Ting, 1997). It can appear, even for each direction of the space, for the case of isotropy or for the case of a CFO2 material with $B_2 = B_3 = 1$ or $A_2 = B_2$ and $A_3 = B_3$, for example. It is shown, however, in Ting and Lee (1997) that the terms q_n can be put into a form which is valid for any case. It is given, for $n = 0, 1, 2$ by:

$$q_{n(1)} = \frac{-1}{2\beta_1\beta_2\beta_3}\left[Re\left\{\frac{p_1^n}{(p_1 - \overline{p_2})(p_1 - \overline{p_3})} - \frac{\delta_{n2}}{3}\right\}\right] \tag{106}$$

and for $n = 3, 4$ by:

$$q_{n(1)} = \frac{-1}{2\beta_1\beta_2\beta_3}\left[Re\left\{\frac{p_1^{n-2}\overline{p_2}\overline{p_3}}{(p_1 - \overline{p_2})(p_1 - \overline{p_3})} - \frac{\delta_{n2}}{3}\right\}\right] \tag{107}$$

with similar expressions for $q_{n(2)}$ and $q_{n(3)}$.

These relations are used in the following to compare the Green tensors related to given elasticity tensors and those related to the closest CFO material.

8. Approximation of orthotropic materials by CFO materials

As shown previously, a CFO4 (or CFO2) material depends on less than nine constants and a randomly chosen orthotropic material is not generally a CFO material. It arises the following question: is it possible to approximate an orthotropic material by a CFO material? This question is in the line of previous research works looking for the best material within a given class to approximate a material which has less symmetries. For example, Pouya and Zaoui (2006) have approximated the elastic properties of different orthotropic materials by elastic properties of materials of higher symmetries. Similarly, Norris obtained the closest elastic tensor of arbitrary symmetry to an elasticity tensor of lower symmetry (Moakher and Norris, 2006), in the sense of different definitions of a distance between different elasticity tensors and solved a similar problem for optimizing an approximation of the acoustic tensor by the acoustic tensor obtained from a material of higher symmetry (Norris, 2006). This problem, studied by Fedorov (1968), leads to the "acoustic distance" defined below. In the following, four distances will be used including the "acoustic distance", because of the use of the acoustic tensor in the process leading to the Green tensor. The properties of these distances may be found in Moakher and Norris (2006) and Norris (2006).

The Euclidean norm is defined by:

$$\|C\|^2 = C_{ijkl}C_{ijkl} \tag{108}$$

It leads to the Euclidean distance defined as:

$$d_E(C_1, C_2) = \|C_1 - C_2\| \tag{109}$$

The log-Euclidean distance d_L and the Riemannian distance d_R are defined by:

$$d_L(C_1, C_2) = \|\log(C_1) - \log(C_2)\| \tag{110}$$

$$d_R(C_1, C_2) = \left\|\log\left(C_1^{-1/2} \cdot C_2 \cdot C_1^{-1/2}\right)\right\| \tag{111}$$

The "acoustic" inner product is defined by:

$$\langle A, B\rangle_a = \mathrm{tr}(I^s AB) \tag{112}$$

where I^s is the "totally symmetric part of the fourth-order identity" defined by $I^s = \frac{1}{3}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj})$.

It leads to the norm:

$$\|A\|_a = \langle A, A\rangle_a^{1/2} \tag{113}$$

Then, the acoustic distance is finally defined by:

$$d_a = \|C_1^* - C_2^*\|_a \tag{114}$$

where the tensor C^* is obtained from the elasticity tensor C_{ijkl} by:

$$C^*_{ijkl} = \frac{1}{2}(C_{ikjl} + C_{iljk}) \tag{115}$$

Obtaining the closest CFO tensor to an arbitrary orthotropic elasticity tensor can be effected by solving the following optimization problem:

For a given initial orthotropic tensor C_{in}, find the tensor C_{cfo} which complies to the set of non-linear compatibility equations for CFO4 (Eqs. (52) and (53)) or for CFO2 (Eqs. (52)–(56), (58), (59)), such that the distance $d(C_{in} - C_{cfo})$ is minimal.

8.1. Optimization using the Euclidean distance

Table 2 shows the relative errors ϵ computed from the Euclidean distance between experimental elasticity tensors of different materials and approximate elasticity tensors by the relative distance given by:

$$\epsilon = \frac{\|C_{app} - C_{exp}\|}{\|C_{exp}\|} \tag{116}$$

The experimental properties are obtained from Kim et al. (1995) and Dieulesaint and Royer (1974) (cited by Pouya and Zaoui (2006) or cited by Tewary (1979) and Tewary (2004)). The computed results comprise the approximate elasticity tensors which are of three different classes: CFO2 and CFO4 classes defined above and TrTI class which corresponds to the transformation by a scaling of the axes of a transversely isotropic material, as defined in Pouya and Zaoui (2006). From the previous sections, it is clear that the TrTI class is of the CFO2 class. All optimisations were performed numerically by using the Euclidean distances. The results show that the approximation is better when using in this order: TrTI, CFO2, CFO4. The relative distance for CFO4 is always less than 10% while for other approximations the relative distance can reach values of the order of 20% or greater.

8.2. Optimization using the different distances

All results using the elastic properties which are optimized by using other distances than the Euclidean distances were compared to the result given by the Euclidean distance for all materials reported in Table 1. The error on the Green tensor was computed as:

$$\epsilon = \max_{i,j}\frac{\max_{\theta,\phi}|G_{app..i,j}(\theta,\phi) - G_{th..i,j}(\theta,\phi)|}{\max_{\theta,\phi}|G_{th..i,j}|(\theta,\phi)} \tag{117}$$

Table 1
Approximate elastic properties ($\times 10$ GPa).

	c_{11}	c_{22}	c_{22}	c_{44}	c_{55}	c_{66}	c_{23}	c_{31}	c_{12}	ϵ
$KB_5O_8 \cdot 4H_2O$										
Experimental	5.82	3.59	2.55	1.64	0.46	0.57	2.31	1.74	2.29	0
TrTI	5.11	3.79	2.55	0.93	1.08	0.89	1.84	2.14	2.61	0.24
CFO2	5.80	4.09	3.12	0.94	0.59	0.67	1.70	1.88	2.15	0.19
CFO4	5.79	3.40	2.21	1.64	0.53	0.60	2.55	1.78	2.28	0.056
S										
Experimental	2.40	2.05	4.83	0.43	0.87	0.76	1.59	1.71	1.33	0
TrTI	2.62	2.13	4.83	0.63	0.69	0.60	1.56	1.73	1.17	0.096
CFO2	2.35	1.87	4.71	0.64	0.96	0.69	1.71	1.77	1.18	0.083
CFO4	2.53	2.22	4.85	0.42	0.89	0.59	1.58	1.71	1.18	0.061
$BaSO_4$										
Experimental	8.8	7.81	10.4	1.17	2.79	2.55	2.89	2.69	4.77	0
TrTI	0.42	7.96	10.4	1.92	2.09	2.15	2.66	2.90	4.37	0.12
CFO2	9.36	7.81	10.2	1.75	1.98	1.99	2.75	2.99	4.57	0.12
CFO4	8.87	8.27	10.1	1.17	3.11	2.08	2.82	3.01	4.40	0.073
PEEK (Polyetheretherketone fiber reinforced composite)										
Experimental	2.85	1.52	1.07	0.22	0.24	0.57	0.76	0.60	0.77	0
CFO2	2.85	1.59	1.01	0.22	0.23	0.56	0.71	0.67	0.73	0.041
CFO4	2.84	1.49	1.04	0.23	0.24	0.58	0.78	0.59	0.78	0.015
TCFC (Tetragonal carbon-fiber composite)										
Experimental	8.03	8.03	141	2.59	2.59	1.81	4.92	4.92	3.11	0
CFO2	7.99	7.99	141	2.59	2.59	2.30	4.90	4.90	3.38	0.0075
CFO4	7.69	7.72	141	2.59	2.58	2.14	4.92	4.92	3.44	0.0065

Table 2
Error on Green tensor for different norms and approximations.

Scheme	Norm	$KB_5O_8 \cdot 4H_2O$	S	$BaSO_4$	PEEK	TFTC
CFO4	Euclidean	0.36	0.054	0.042	0.006	0.033
	Acoustic	0.35	0.075	0.046	0.023	0.033
	Log-Euclidean	0.16	0.060	0.041	0.021	0.021
	Riemannian	0.11	0.069	0.041	0.019	0.019
CFO2	Euclidean	0.21	0.10	0.096	0.039	0.066
	Acoustic	0.20	0.087	0.069	0.031	0.063
	Log-Euclidean	0.19	0.082	0.053	0.045	0.045
	Riemannian	0.19	0.086	0.052	0.054	0.045

where $G_{app.,i,j}$ and $G_{th.,i,j}$ correspond to the approximate and theoretical values of Green tensor.

These values are computed for values of θ and ϕ (30 values for each angle) ranging between 0 and $\frac{\pi}{2}$. The table shows that the CFO4 approximation leads, except in the first case for the Euclidean norm, to a better approximation than the CFO2 approximation. Results for the Log-Euclidean distance and the Riemanian distance are comparable but can differ significantly from the results obtained from the Euclidean norm and the acoustic norm.

The use of a convenient norm and of the CFO4 approximation leads to a relative error which is less than 11% in any case and is less than 5% for the last three cases.

8.3. *Comparison of exact and approximate Green tensors*

Figs. 1, 3 and 5 show the variation of the components G_{11}, G_{22}, G_{12} of the Green tensor as a function of θ and ϕ for the first material computed from the exact solution, while Figs. 2, 4, 6 show the same components obtained from the approximation "CFO4". All figures show a good agreement between exact and approximate values.

9. Synthesis and conclusion

Obtaining a closed form expression of the Green tensor for an elastic orthotropic medium is possible in the most general case only for some specific directions between source and reception point. It is due to the requirement, in the general case, to solve a sixth order equation for obtaining the poles of a rational fraction and to perform the integration by residue calculus.

Such a difficulty led to numerous developments to perform numerical integrations and obtain expressions for the derivatives allowing to compute the stress tensor induced by a point source.

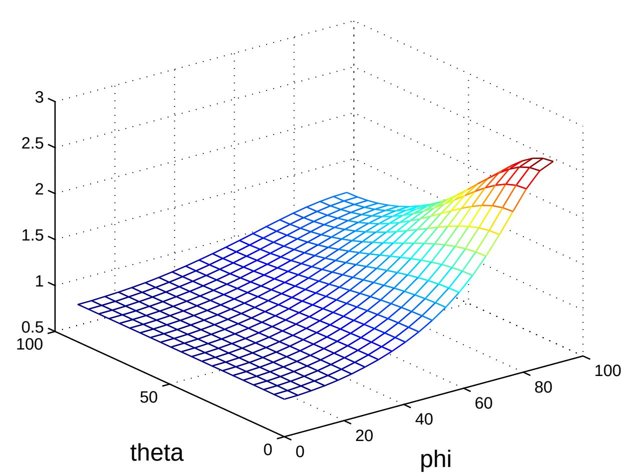

Fig. 1. G_{11} component of the Green tensor as a function of θ and ϕ: exact value.

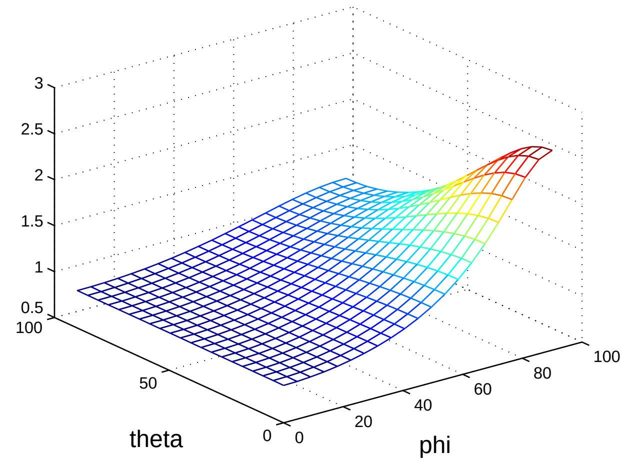

Fig. 2. G_{11} component of the Green tensor as a function of θ and ϕ: CFO4 approximation.

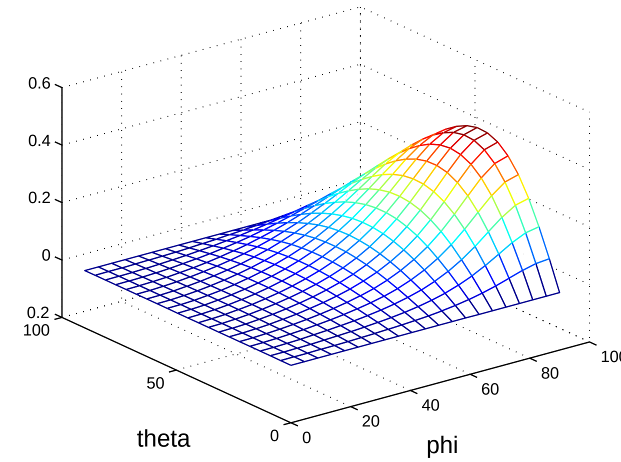

Fig. 5. G_{12} component of the Green tensor as a function of θ and ϕ: exact value.

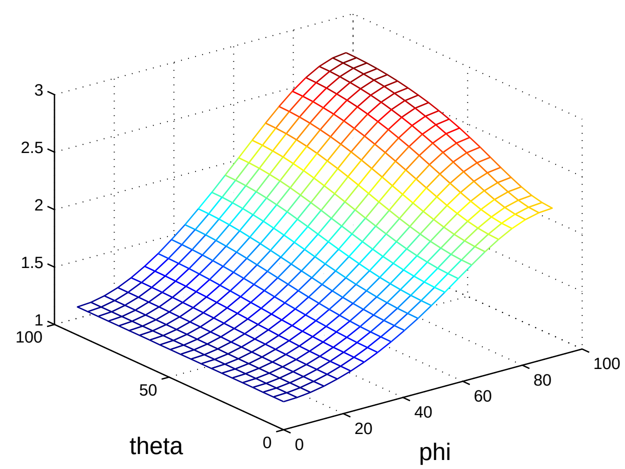

Fig. 3. G_{22} component of the Green tensor as a function of θ and ϕ: exact value.

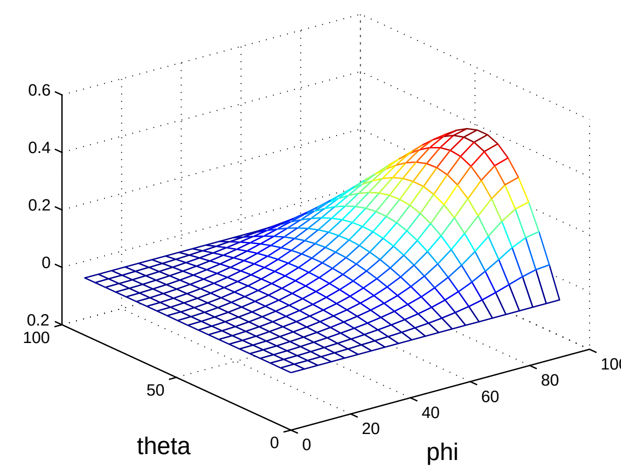

Fig. 6. G_{12} component of the Green tensor as a function of θ and ϕ: CFO4 approximation.

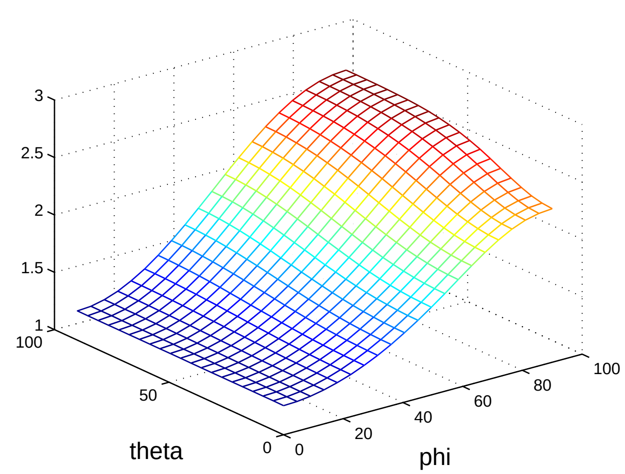

Fig. 4. G_{22} component of the Green tensor as a function of θ and ϕ: CFO4 approximation.

In the present paper, it was shown that a closed form solution can be obtained for specific orthotropic elastic media (of "CFO2" and "CFO4" classes), and that the solution which is obtained is an extension of the classical solution for a transversely isotropic material.

Two different cases were studied: the case of elastic properties of "CFO4" class such that the process of computation of the eigenvalues needs, for each direction of the space, the solution of a quadratic equation and of a fourth order equation, and the "CFO2" class such that the process needs the solution of three quadratic equations. The CFO2 class contains the "TrTI" class of materials (obtained from a transversely isotropic material by a scaling of the axes) introduced by Pouya and Zaoui (2006).

It is shown that transversely isotropic materials are of CFO2 class, that tetragonal and orthotropic materials of CFO2 class are necessarily transversely isotropic and that cubic materials of CFO4 class are necessarily isotropic. In addition, it is shown that any CFO2 or CFO4 material is such that there exists a scaling of the axes such that there is a constant $p = i$ eigenvalue for any direction between source and reception point.

The approximation of five different experimental orthotropic elasticity tensors by elasticity tensors of CFO2 and CFO4 classes shows that the best approximation is obtained by the CFO4 class, the level of relative difference between approximate and exact elasticity tensors being inferior to 11%, on condition that the best norm is used to obtain the approximate elasticity tensor, the norm being chosen between Euclidean norm, "acoustic norm", Log-Euclidean norm and Riemannian norm.

A comparison between three components of the Green tensor for all directions of the space obtained from the "CFO4" approximation and from the exact solution in the case of the PEEK material shows that the components obtained from the approximation compare well with the exact solution.

Finally, having obtained (approximate) expressions of the Green tensor which use closed form expressions of the eigenvalues makes it possible to obtain directly the derivatives of the Green tensor, which can be used within the framework of the boundary element method, for example, or for the computation of the stress field induced by concentrated forces.

Appendix A. Coefficients of the determinant of the matrix related to the acoustic tensor for an orthotropic material

$$\begin{aligned}
a_{111} &= c_{11}c_{66}c_{55}\\
a_{222} &= c_{66}c_{22}c_{44}\\
a_{333} &= c_{55}c_{44}c_{33}\\
a_{112} &= c_{11}(c_{22}c_{55}+c_{44}c_{66})+c_{55}(c_{66}^2-c_{12}^{*2})\\
a_{113} &= c_{11}(c_{33}c_{66}+c_{44}c_{55})+c_{66}(c_{55}^2-c_{31}^{*2})\\
a_{221} &= c_{22}(c_{11}c_{44}+c_{55}c_{66})+c_{44}(c_{66}^2-c_{12}^{*2})\\
a_{223} &= c_{22}(c_{33}c_{66}+c_{55}c_{44})+c_{66}(c_{44}^2-c_{23}^{*2})\\
a_{331} &= c_{33}(c_{11}c_{44}+c_{66}c_{55})+c_{44}(c_{55}^2-c_{31}^{*2})\\
a_{332} &= c_{33}(c_{22}c_{55}+c_{66}c_{44})+c_{55}(c_{44}^2-c_{23}^{*2})\\
a_{123} &= c_{11}c_{22}c_{33}+c_{11}c_{44}^2+c_{22}c_{55}^2+c_{33}c_{66}^2+2c_{44}c_{55}c_{66}\\
&\quad +2c_{12}^*c_{23}^*c_{31}^*-c_{12}^{*2}c_{33}-c_{23}^{*2}c_{11}-c_{31}^{*2}c_{22}
\end{aligned} \tag{A.1}$$

where

$$\begin{aligned}
c_{12}^* &= c_{12}+c_{66}\\
c_{23}^* &= c_{23}+c_{44}\\
c_{31}^* &= c_{31}+c_{55}
\end{aligned} \tag{A.2}$$

The components d_{ijk} are given by:

$$d_{ijk} = a_{ijk}/a_{111} \tag{A.3}$$

Appendix B. Coefficients of the determinant of the matrix related to the acoustic tensor for a transversely isotropic material

$$\begin{aligned}
a_{111} &= \frac{1}{2}c_{11}c_{44}(c_{11}-c_{12})\\
a_{333} &= c_{44}^2c_{33}\\
a_{112} &= \frac{3}{2}c_{44}c_{11}(c_{11}-c_{12})\\
a_{113} &= \frac{1}{2}(c_{11}^2c_{33}-c_{11}c_{12}c_{33}-c_{13}^2c_{11}+c_{13}^2c_{12})+c_{11}c_{44}^2\\
&\quad -c_{13}c_{11}c_{44}+c_{13}c_{12}c_{44}\\
a_{331} &= \frac{1}{2}c_{44}(3c_{11}c_{33}-c_{12}c_{33}-2c_{13}^2-4c_{13}c_{44})\\
a_{123} &= 2c_{11}c_{44}^2+c_{11}^2c_{33}-c_{11}c_{12}c_{33}-c_{13}^2c_{11}+c_{13}^2c_{12}\\
&\quad -2c_{13}c_{11}c_{44}+2c_{13}c_{12}c_{44}
\end{aligned} \tag{B.1}$$

Appendix C. Coefficients of the determinant of the matrix related to the acoustic tensor for a tetragonal material

$$\begin{aligned}
a_{111} &= c_{11}c_{44}c_{66}\\
a_{333} &= c_{44}^2c_{33}\\
a_{112} &= c_{44}\left(c_{66}^2+c_{11}^2+c_{11}c_{66}-c_{12}^{*2}\right)\\
a_{113} &= c_{11}c_{44}^2+c_{11}c_{66}c_{33}+c_{66}c_{44}^2-c_{13}^{*2}c_{66}\\
a_{331} &= c_{44}\left(c_{11}c_{33}+c_{66}c_{33}+c_{44}^2-c_{13}^{*2}\right)\\
a_{123} &= c_{11}^2c_{33}+2c_{11}c_{44}^2+2c_{66}c_{44}^2+c_{33}c_{66}^2+2c_{12}^*c_{23}^{*2}-c33c_{12}^{*2}-2c_{11}c_{13}^{*2}
\end{aligned} \tag{C.1}$$

Appendix D. Matrix related to the acoustic tensor in the orthotropic case as a function of c_{ij}

The matrix related to the acoustic tensor for an orthotropic elasticity tensor when using symmetry axes is given by:

$$\mathbf{Q} = \begin{bmatrix} c_{11}k_1^2+c_{66}k_2^2+c_{55}k_3^2 & k_1k_2(c_{12}+c_{66}) & k_3k_1(c_{13}+c_{55})\\ k_1k_2(c_{12}+c_{66}) & c_{22}k_2^2+c_{44}k_3^2+c_{66}k_1^2 & k_2k_3(c_{23}+c_{44})\\ k_3k_1(c_{13}+c_{55}) & k_2k_3(c_{23}+c_{44}) & c_{33}k_3^2+c_{55}k_1^2+c_{44}k_2^2 \end{bmatrix}$$

A checking of this matrix can be performed by using a vector $\mathbf{n}$ in the plane (x_1x_2) and the components of $\mathbf{n}$ as: $(\cos(\phi), 0, \sin(\phi)) = (c, 0, -s)$. This leads to the matrix $\mathbf{Q}_0$ of Eq. (9) given by:

$$\mathbf{Q}_0 = \begin{bmatrix} c_{11}\cdot c^2+c_{55}\cdot s^2 & 0 & -c\cdot s\cdot(c_{13}+c_{55})\\ 0 & c_{66}\cdot c^2+c_{44}\cdot s^2 & 0\\ -c\cdot s\cdot(c_{13}+c_{55}) & 0 & c_{55}\cdot c^2+c_{33}\cdot s^2 \end{bmatrix}$$

This result is the same as the one given by Ting and Lee (1997, Eq. (4.5)) for the case of an orthotropic material.

References

Chang, C., Chang, Y., 1995. Green's function for elastic medium with general anisotropy. ASME Journal of Applied Mechanics 62, 573–578.

Condat, M., Kirchner, M., 1987. Computational anisotropic elasticity. Physica Status Solidi (B) 144, 137–143.

Dahan, M., Predeleanu, M., 1980. Solutions fondamentales pour un milieu élastique à isotropie transverse. Journal of Applied Mathematics and Physics 31, 413–424.

Dederichs, P., Leibfried, G., 1969. Elastic Green's function of anisotropic cubic crystals. Physical Review 188 (3), 1175–1183.

De Saint Venant, B., 1863. Sur la distribution desélasticités autour de chaque point d'un solide ou d'un milieu de contexture quelconque, particulièrement lorsqu'il est amorphe sansêtre isotrope. Journal de Mathématiques Pures et Appliquées 69, 247–254.

Dieulesaint, E., Royer, D., 1974. Ondes Tlastiques dans les cristaux. Masson, Paris.

Faux, D., Pearson, G., 2000. Green's tensors for anisotropic elasticity: application to quantum dots. Physical Review B 62 (8), 4798–4801.

Fedorov, F., 1968. Theory of Elastic Waves in Crystals. Plenum Press, New York.

Head, A., 1979. The Galois unsolvability of the sextic equation of anisotropic elasticity. Journal of Elasticity 9, 9–20.

Kim, K., Ohtani, T., Baker, A., Sachse, W., 1995. Determination of all elastic constants of orthotropic plate specimens from group velocity data. Research in Nondestructive Evaluation 7, 13–29.

Kroner, E., 1953. Das fundamentalintegral der anisotropen elastischen differentialgleichungen. Zeitschrift fur Physik 136, 402–410.

Kuznetsov, S., 1996. Some singular solutions of the theory of elasticity. Journal of Applied Mathematics and Mechanics 60 (5), 863–865.

Lee, V., 2002. Evaluation of the three-dimensional elastic Green's function in anisotropic cubic media. International Journal of Engineering Science 40, 1349–1361.

Lee, V., 2003. Explicit expression of the derivatives of elastic Green's functions for general anisotropic materials. Mechanics Research Communications 30, 241–249.

Lejcek, L., 1969. The Green function of the theory of elasticity in an anisotropic hexagonal medium. Czech Journal of Physics B19, 799–803.

Lifshitz, I., Rozenzweig, L., 1947. On the construction of the Green's tensor for the basic equation of the theory of elasticity of an anisotropic infinite medium. Zhurnal Experimentalnoi i Theoreticheskoi Fiziki 17, 783–791.

Moakher, M., Norris, A., 2006. The closest elastic tensor of arbitrary symmetry to an elastic tensor of lower symmetry. Journal of Elasticity 85, 215–263.

Mura, T., 1987. Micromechanics of Defects in Solids. Martinus-Nijhoff, Dordrecht.

Mura, T., Kinoshita, N., 1971. Green's functions for anisotropic elasticity. Physica Status Solidi (B) (47), 607–618.

Nakamura, G., Tanuma, K., 1997. A formula for the fundamental solution of anisotropic elasticity. Quarterly Journal of Mechanics and Applied Mathematics 50 (2), 179–194.

Norris, A., 2006. Elastic moduli approximation of higher symmetry for the acoustical properties of an anisotropic material. Journal of the Acoustical Society of America 119 (4), 2114–2121.

Pan, Y., Chou, T., 1976. Point force solution for an infinite transversely isotropic solid. ASME Journal of Applied Mechanics 43, 608–612.

Pouya, A., 2007a. Fonction de Green pour les matériaux à anisotropie ellipsoidale. Compte Rendu de l'Académie des Sciences 335, 407–413.

Pouya, A., 2007b. Green's function solution and displacement potentials for Transformed Transversely Isotropic materials. European Journal of Mechanics A: Solids 26, 491–502.

Pouya, A., Zaoui, A., 2006. A transformation of elastic boundary value problems with application to anisotropic behaviour. International Journal of Solids and Structures 43, 4937–4956.

Sales, M., Gray, L., 1998. Evaluation of the anisotropic Green's function and its derivatives. Computers and Structures 69, 247–254.

Tewary, V., 1979. Mechanics of Fiber Composites. Wiley, New York.

Tewary, V., 2004. Elastostatic Green's function for advanced materials subject to surface loading. ASCE Journal of Engineering Mechanics 49, 289–304.

Ting, T., 1996. Anisotropic Elasticity. Oxford University Press, Oxford.

Ting, T., Lee, V., 1997. The three-dimensional elastostatic Green's function for general anisotropic linear elastic solids. Quarterly Journal of Mechanics and Applied Mathematics 50 (3), 407–426.

Tonon, F., Pan, E., Amadei, B., 2001. Green's functions and boundary element method formulation for 3D anisotropic media. Computers and Structures 79, 469–482.

Wang, C., 1997. Elastic field produced by a point source in solids of general anisotropy. ASCE Journal of Engineering Mechanics 32, 41–52.

Wang, Y., Ting, T., 1997. The Stroh formalism for anisotropic materials that possess an almost extraordinary degenerate matrix. International Journal of Solids and Structures 34 (4), 401–413.

Willis, J., 1969. The Green's function of the theory of elasticity in an anisotropic hexagonal medium. Czech Journal of Physics 19B, 799–803.